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# Adjoint Operators of Fourier-Laplace Transform

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**ABSTRACT**

The concept of Fourier Transformation and Laplace Transformation play a vital role in diverse areas of Science and technology such as electric analysis, communication engineering, control engineering, linear system, analysis, statistics, optics, quantum physics, solution of partial differential operation etc. These Fourier and Laplace Transforms have various properties and these Properties have opened up a variety of applications. This paper provides the Generalization of Fourier-Laplace Transform in the Distributional sense and described some Adjoint Operators of Fourier-Laplace Transform.



## 1. INTRODUCTION

Integral transforms are frequently applied when solving differential and integral equations and their choice depends on the type of considered equation. The main condition when choosing an integral transform is the possibility to reduce a differential or integral expression to a simpler differential equation. It is obvious that the number of integral transforms can be considerably increased by introducing new kernels [1]. Fourier and Laplace transform technique is applicable in many field of science and technology such as control Engineering, Communication, Signal analysis and Design, system analysis, solving differential equations as well as in Medical field [2].

We reported various properties like linearity, Shifting, Scaling, Convolution, Differentiation and Integration of Fourier and Laplace transform elsewhere and these properties opened up a variety of applications. The Shifting property of Fourier transform identifies the fact that a linear displacement in time corresponds to a linear phase factor in the frequency domain. This becomes useful and important when we discuss filtering and the effects of the phase characteristic of a filter in the time domain. The differentiation property for Fourier Transform is very useful. In the time domain we recognize that differentiation will emphasize these abrupt changes, and this property states that consistent with this result, the high frequencies are amplified in relation to the low frequencies [3]. By all the above properties of Fourier and Laplace transform we can solve various problems like heat equation, wave equation etc. [4], [5].

Testing function spaces are needed to develop Fourier-Laplace transform, therefore these are defined along with distribution as follows:

### 1.1 THE SPACE $FL_{a,b,\alpha}$

This space is given by

$$FL_{a,b,\alpha} = \left\{ \phi : \phi \in E_+ \,/\, \xi_{a,b,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^k e^{ax} D_t^l D_x^q \phi(t,x) \right| \le C_{lq} A^k k^{k\alpha} \right\} \qquad (1.1.1)$$

where the constants $A$ and $C_{lq}$ depend on the testing function $\phi$.

### 1.2 THE SPACE $FL_{a,b,\gamma}$

It is given by

$$FL_{a,b,\gamma} = \left\{ \phi : \phi \in E_+ \,/\, \gamma_{a,b,k,q,l}\phi(t,x) = \sup_{\substack{0<t<\infty \\ 0<x<\infty}} \left| t^k e^{ax} D_t^l D_x^q \phi(t,x) \right| \le C_{lk} A^q q^{q\gamma} \right\} \qquad (1.2.1)$$

where, $k,l,q = 0,1,2,3,........$ and the constants depend on the testing function $\phi$ .

### 1.3. DISTRIBUTIONAL GENERALIZED FOURIER-LAPLACE TRANSFORMS (*FLT*)

For $f(t,x) \in FL_{a,\alpha}^{*\beta}$ , where $FL_{a,\alpha}^{*\beta}$ is the dual space of $FL_{a,\alpha}^{\beta}$ . It contains all distributions of compact support. The distributional Fourier-Laplace transform is a function of $f(t,x)$ and is defined as

$$FL\{ f(t,x)\} = F(s,p) = \left\langle f(t,x), e^{-i(st-ipx)} \right\rangle, \qquad (1.3.1)$$

where, for each fixed $t \; (0<t<\infty)$, $x \; (0<x<\infty)$, $s>0$ and $p>0$, the right hand side of (1.3.1) has a sense as an application of $f(t,x) \in FL_{a,\alpha}^{*\beta}$ to $e^{-i(st-ipx)} \in FL_{a,\alpha}^{\beta}$ .

This paper provides the generalization of Fourier-Laplace transform in the distributional sense and defined some Adjoint Operators of Fourier-Laplace transforms. The summary of this paper is as follows:

In section 2, Adjoint shifting operator for Fourier-Laplace transform is defined, Adjoint diffetrential operators of Fourier-Laplace transforms is described in section 3. Lastly conclusions are given in section 4.

Notations and terminology as per Zemanian. [6], [7].

## 2. ADJOINT OPERATORS OF FOURIER-LAPLACE TRANSFORM

### 2.1. Theorem:

The adjoint shifting operator is a continuous function from $FL_{a,b,\alpha}^{*}$ to $FL_{a,b,\alpha}^{*}$. The adjoint operator $f(t,x) \to f(t-\tau,x)$ leads to the operation transform formula

$$FL\{ f(t-\tau,x)\} = e^{-is\tau} FL\{ f(t,x)\}$$

**Proof:** Consider,

$$\begin{aligned} FL\{ f(t-\tau,x)\} &= \left\langle f(t-\tau,x) e^{-i(st-ipx)} \right\rangle \\ &= \left\langle f(t,x), e^{-is(t+\tau)} e^{-px} \right\rangle \\ &= \left\langle f(t,x), e^{-i[s(t+\tau)-ipx]} \right\rangle \\ &= e^{-is\tau} \left\langle f(t,x), e^{-i(st-ipx)} \right\rangle \\ &= e^{-is\tau} FL\{ f(t,x)\} \end{aligned}$$

$$\therefore FL\{ f(t-\tau,x)\} = e^{-is\tau} FL\{ f(t,x)\}$$

**2.2. Theorem:**

The adjoint shifting operator is a continuous function from $FL^{*}_{a,b,\alpha}$ to $FL^{*}_{a,b,\alpha}$. The adjoint operator $f(t,x)\rightarrow f(t,x-q)$ leads to the operation transform formula $FL\{f(t,x-q)\}=e^{-pq}FL\{f(t,x)\}$

**Proof:-**Consider,

$$FL\{f(t,x-q)\}=\left\langle f(t,x-q),e^{-i(st-ipx)}\right\rangle$$
$$=\left\langle f(t,x),e^{-i[st-ip(x+q)]}\right\rangle$$
$$=e^{-pq}\left\langle f(t,x),e^{-i(st-ipx)}\right\rangle$$
$$=e^{-pq}FL\{f(t,x)\}$$

$$\therefore FL\{f(t,x-q)\}=e^{-pq}FL\{f(t,x)\}$$

**2.3. Proposition:**

The adjoint shifting operator is a continuous function from $FL^{*}_{a,b,\alpha}$ to $FL^{*}_{a,b,\alpha}$. The adjoint operator $f(t,x)\rightarrow f(t-\tau,x-q)$. Correspondingly we can prove $FL\{f(t-\tau,x-q)\}=e^{-i(s\tau-ipq)}FL\{f(t,x)\}$. Note that Fourier-Laplace transform is shift-shift invariant.

**3. ADJOINT DIFFERENTIAL OPERATORS OF FOURIER-LAPLACE TRANSFORM**

**3.1. Theorem:**

The adjoint differential operator $f\rightarrow D_t f$ is continuous linear mapping from the dual space $FL^{*}_{a,b,\alpha}$ into itself. Corresponding transform formula is $FL\{D_t f(t,x)\}=(is)FL\{f(t,x)\}$

**Proof:** Consider,

$$FL\{D_t f(t,x)\}=\left\langle D_t f(t,x),e^{-i(st-ipx)}\right\rangle$$
$$=\left\langle f(t,x),-D_t e^{-i(st-ipx)}\right\rangle$$
$$=\left\langle f(t,x),(is)e^{-i(st-ipx)}\right\rangle$$
$$=(is)\left\langle f(t,x),e^{-i(st-ipx)}\right\rangle$$
$$\therefore FL\{D_t f(t,x)\}=(is)FL\{f(t,x)\}$$

**3.2. Theorem:**

The adjoint differential operator $f\rightarrow D_x f$ is continuous linear mapping from the dual space $FL^{*}_{a,b,\alpha}$ into itself. Corresponding transform formula is $FL\{D_x f(t,x)\}=pFL\{f(t,x)\}$

**Proof:** Consider,

$$FL\{D_x f(t,x)\}=\left\langle D_x f(t,x),e^{-i(st-ipx)}\right\rangle$$
$$=\left\langle f(t,x),-D_x e^{-i(st-ipx)}\right\rangle$$
$$=\left\langle f(t,x),pe^{-i(st-ipx)}\right\rangle$$
$$=p\left\langle f(t,x),e^{-i(st-ipx)}\right\rangle$$
$$\therefore FL\{D_x f(t,x)\}=pFL\{f(t,x)\}$$

**3.3. Theorem:**

The adjoint operator $f \to \theta f$ is a continuous linear mapping of $FL^{*}_{a,b,\alpha}$ into itself, the adjoint operator $f(t,x) \to e^{-i(\tau t - i\alpha x)} f(t,x)$ corresponding operator transform formula is $FL\left\{e^{-i(\tau t - i\alpha x)} f(t,x)\right\} = F(s+\tau, p+\alpha)$.

**Proof:** Consider,

$$FL\left\{e^{-i(\tau t - i\alpha x)} f(t,x)\right\} = \left\langle e^{-i(\tau t - i\alpha x)} f(t,x), e^{-i(st - ipx)} \right\rangle$$

$$= \left\langle f(t,x), e^{-it(s+\tau)} e^{-x(p+\alpha)} \right\rangle$$

$$= \left\langle f(t,x), e^{-i\left[(s+\tau)t - i(p+\alpha)x\right]} \right\rangle$$

$$= F(s+\tau, p+\alpha)$$

$$\therefore FL\left\{e^{-i(\tau t - i\alpha x)} f(t,x)\right\} = FL\left\{f(t,x)\right\}(s+\tau, p+\alpha) = F(s+\tau, p+\alpha)$$

**3.4. Theorem:**

Noting above proposition the adjoint operator is $f(t,x) \to (-it)^{k_1} (-x)^{k_2} f(t,x)$. Corresponding operator transform formula is $FL\left\{(-it)^{k_1} (-x)^{k_2} f(t,x)\right\} = D_s^{k_1} D_p^{k_2} F(s,p)$.

**Proof:** Consider,

$$FL\left\{(-it)^{k_1} (-x)^{k_2} f(t,x)\right\} = \left\langle (-it)^{k_1} (-x)^{k_2} f(t,x), e^{-i(st - ipx)} \right\rangle$$

$$= \left\langle f(t,x), (-it)^{k_1} (-x)^{k_2} e^{-i(st - ipx)} \right\rangle$$

$$= D_s^{k_1} D_p^{k_2} F(s,p)$$

$$\therefore FL\left\{(-it)^{k_1} (-x)^{k_2} f(t,x)\right\} = D_s^{k_1} D_p^{k_2} F(s,p)$$

## 4. CONCLUSION

This paper presents the Generalization of Fourier-Laplace transform in the distributional sense. And some Adjoint Operators of Fourier-Laplace transform along with the properties of Fourier-Laplace transform are defined, which will be useful when this transform will be used to solve differential and integral equations.

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