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APPLICATIONS AND PROPERTIES OF CONDITIONAL CHROMATIC NUMBER OF GRAPHS

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ABSTRACT

This paper studies the concepts of Applications and properties of conditional chromatic number of graphs. The main results are

1) *results on chromatic conjecture*
2) *Seven properties of $\chi(G, \neg C_j)$*
3) *Implications for $\chi(G, \neg C_j)$ from $\chi(G,P_a)$*

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1. Conditional Chromatic number

1.1. Introduction:

The generalized chromatic number is defined as the minimum number of colors needed to color the nodes of G such that every subgraph of G induced by a color class has property P [1]. This generalization to any property is now referred to as the conditional chromatic number of a graph [5]. In the following sections we will discuss the history of the conditional chromatic number, relate known results to the property of containing no cycles, and prove new results for the value of the conditional chromatic number with respect to the property of forbidding cycles of a fixed length.

1.2. Results on Chromatic Conjecture

This chapter begins the study of the conditional chromatic number with respect to property, $P= \neg G$, for a fixed $j \geq 3$. The first theorem states seven basic properties of $\chi(G, \neg C_j)$ that follow from definitions. These properties will be used throughout the remainder of the paper. In the second section of this chapter, established conditional chromatic number results for point of orbitrary are used to prove bounds for the value of $\chi(G, \neg C_j)$. In the third section, general conditional chromatic number results for properties that are hereditary are restated specifically for $\chi(G, \neg C_j)$.

2. Properties

2.1. Seven properties of $\chi(G, \neg C_j)$

Although all of the properties in this theorem are straightforward, they will be important to the establishment of new results. Note that properties *(b)*, *(d)*, *(f)*, and *(g)* follow from observations.

2.2. Theorem

(a) $\chi(K_n, \neg C_j) = \left\lceil \frac{n}{(j-1)} \right\rceil$.

(b) If H is a subgraph of the graph G, then $\chi(H, \neg C_j) \leq \chi(G, \neg C_j)$

(c) For every graph G, $\chi(G, \neg C_j) \leq \left\lceil \frac{n}{j-i} \right\rceil$

(d) If G is disconnected with components

$G_1, G_2, \ldots, G_r$, *then* $\chi(G, \neg C_j) = \max_i \{\chi(G_i, \neg C_j)\}$

(e) $\chi(G, \neg C_j) = 1$ if and only if G contains no Cj, Consequently, for $n < j$,

$\chi(G, \neg C_j) = 1$.

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(f) $\chi(G, \neg C_j) \leq \chi(G,P_a) \leq \chi(G)$.

(g) For the complete bipartite graph $G= K_{p,q}$ with $p \leq q$,

$$\chi(G, \neg C_j) = \begin{cases} 1 \ \text{if } j \text{ is odd or } j > 2p \\ 2 \ \text{if } j \text{ is even and } 4 \leq j \leq 2p \end{cases}$$

Proof.

(a) A proper coloring of K_n will consist of color classes $V_1 \dots V_k$ such that each $\langle V_i \rangle$ is a subgraph of K_n. Each induced subgraph of a complete graph is complete. Therefore, every color class that consists of more than $j-1$ vertices will contain an induced C_j. Every color class that consists of $j-1$ vertices or fewer will not contain a C_j. This implies the maximum number of vertices in each color class is $j-1$. Therefore the total number of color classes needed to color K_n is

$$\left\lceil \frac{n}{(j-1)} \right\rceil = \chi(K_n, \neg C_j).$$

(b) If G is complete the result follows from (a) above. Otherwise, from the definition of $\chi(G)$ we know that $\chi(G, \neg C_j)$ is the minimum number of color classes needed to properly color the vertices of G. Since H is a subgraph of G, coloring the vertices of H as we did with G will yield a proper coloring of H. Since H may contain fewer than $\chi(G, \neg C_j)$ colors. In any case $\chi(H, \neg C_j)$ can be colored with $\chi(G, \neg C_j)$ of fewer colors. Therefore $\chi(H, \neg C_j) \leq \chi(G, \neg C_j)$.

(c) Every graph G is either complete, or the subgraph of a complete graph. If G is complete the result follows by (a) above, if G is not complete, then the result follows by (b) above.

(d) Each G_i is a subgraph of G so by (b) above $max_i \{\chi(G_i, \neg C_j)\} \leq \chi(G, \neg C_j)$. We now must show $max_i \{\chi(G_i, \neg C_j)\} \geq \chi(G, \neg C_j)$. Assume

(e) $max_i \{\chi(G_i, \neg C_j)\} = k$. Then every other component of G can be properly colored with $\leq k$ colors. Since all components are disconnected from each other, the vertices from each non-maximal component can be included in the k color classes of the maximal component without creating a C_j. Then the number of colors needed to color G is no more than $k=max\{\chi(G_i, \neg C_j)\}$.

(f) $\chi(G, \neg C_j) = 1 \Leftrightarrow$ every vertex of G can be colored the same color $\Leftrightarrow G$ contains no C_j if $n < j$ then G cannot contain a C_j. Following the argument backwards implies $\chi(G, \neg C_j) = 1$.

(g) It follows from the definitions of $\chi(G)$ and $\chi(G,P_a)$ that a coloring that satisfies the former, will satisfy the latter, and a coloring that satisfies the latter will not necessarily satisfy the former, therefore $\chi(G,P_a) \leq \chi(G)$. Similarly, $\chi(G, \neg C_j) \leq \chi(G, P_a)$.

(h) If G is the complete bipartite graph, then G contains no odd cycles but contains even cycles of every length j for j up to and including $2p$. The graph can be colored with two colors by assigning a color to each of the two sets of the bipartition. □

2.3. Implications for $\chi(G, \neg C_j)$ from $\chi(G,P_a)$

In general the conditional property P_a that each color class have no cycles. The P_a-chromatic number is also called the point arboricity. From Theorem 2.2 (f) it follows that an upper bound on the point arboricity for a family of graphs is also an upper bound for the $\neg C_j$-chromatic numbers for the given family and each value of $j \geq 3$. This section examines the major results of P_a-chromatic numbers and translates them into $\neg C_j$-chromatic number results.

The P_a-chromatic number results of this section are divided into three parts, the first examines upper bounds for the P_a-chromatic number, the second improves these results if G is planar (or outerplanar), and the third consists of more recent P_a-chromatic number results.

The study of P_a-chromatic numbers began with a series of papers in 1968 in which the authors proved upper bounds for $\chi(G,P_a)$. All of the results below that are attributed to [4] and [3] can be similarly attributed to [7].

Chartrand, Kronk and wall [4] interpreted the point arboricity (P_a) of a graph as a "coloring number" since $P_a(G)$ is the minimum number of colors needed to color the vertices of G such that no cycle has all of its vertices colored the same. Using an inductive argument on the number of vertices of a graph they proved the following theorem.

2.4. Theorem *For any graph G.*

$$\chi(G, P_a) \leq \left\lceil \frac{1 + \Delta(G)}{2} \right\rceil$$

For complete graph Theorem is tight because the maximum number of vertices in any color class is two-anymore and the graph induced by the vertices of the color class consists of a cycle of length three. Therefore,

$$\chi(K_n, P_a) = \left\lceil \frac{n}{2} \right\rceil = \left\lceil \frac{1+(n-1)}{2} \right\rceil = \left\lceil \frac{1+\Delta(G)}{2} \right\rceil$$

On the other hand, for planar graphs with large maximum degree this bound is not very strong. For example the complete bipartite graph $K_{1,10}$ is planar with $\Delta(G)=10$, so from Theorem 2.4. $\chi(G,P_a) \leq \left\lceil \frac{1+10}{2} \right\rceil = 6$. But $K_{1,10}$ contains no cycles so $\chi(G, P_a)$ =1. As we will see below this bound can be improved for planar graphs.

2.5. Theorem: *For any graph G,*

$$\chi(G, P_a) \leq 1 + \left\lceil \frac{\Delta(G)}{2} \right\rceil$$

The proof of theorem 2.5. relies on the fact that there exists an induced *n*-critical subgraph *H* of *G*. Additionally, the minimum vertex degree of *H* is less than or equal to the maximum over all minimum degrees of subgraphs of *H*. Using the *n*-critical property the result follows, see [3] for details. Note that theorem 2.3 and 2.4 produce the same bound when $\Delta(G)$ is even, but when $\Delta(G)$ is odd the bound in theorem 2.4 is smaller.

The next theorem improves the upper bound for $\chi(G,P_a)$ in the case that *G* is non-regular, i.e., every vertex is not of the same degree.

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